

Geometric Engineering of $N=1$ Quantum Field Theories

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We construct local geometric model in terms of F- and M-theory compactification on Calabi-Yau fourfolds which lead to $N = 1$ Yang-Mills theory in $d = 4$ and its reduction on a circle to $d = 3$. We compute the superpotential in $d = 3$, as a function of radius, which is generated by the Euclidean 5-brane instantons. The superpotential turns out to be the same as the potential for affine Toda theories. In the limit of vanishing radius the affine Toda potential reduces to the Toda potential.

1. Geometric Engineering

One of the most powerful consequences of our deeper understanding of the dynamics of string theory has been the appreciation of the fact that gauge dynamics can be encoded geometrically in the structure of compactifications of type II superstrings (see e.g. [1][2][3][4][5][6][7]). Gauge groups arise through ADE singularities of geometry (and their fibrations [8][9]), whereas matter arises as loci of enhanced singularities [10]. This not only leads to a unified description of gravitational and gauge theory dynamics, but it also leads directly to a deeper understanding of gauge dynamics, even in the limit of turning off gravitational effects [11][12][13]. The basic idea is to *geometrically engineer* the gauge symmetry and matter content one is interested in, and then study the corresponding theory using string techniques. In particular in [13] it was shown how to engineer $N = 2$ theories in $d = 4$. In particular if one is interested in studying $N = 2$ theories with $G = SU(N)$ with N_f flavors, one looks for a geometry where over a \mathbf{P}^1 there is an A_{N-1} singularity and where over N_f points on \mathbf{P}^1 the singularity enhances to A_N . Moreover this reduces the computation of the prepotential in the $N = 2$ field theories to the question of worldsheet instantons of type IIA strings, which is computable using (local) mirror symmetry. In particular the contribution of spacetime instantons of gauge theory to the prepotential are mapped to the growth of the number of worldsheet instantons in a particular configuration.

The aim of this note is to initiate a study of $N = 1$ theories in $d = 4$ along the same lines and their reduction to $d = 2, 3$. Our aim here is to study the case with no matter; the case with matter can be done in a similar way, and the results will be presented elsewhere [14].

2. $N = 1$ Yang-Mills in $d = 4$ and F-theory on CY 4-fold

If one compactifies F-theory on CY 4-folds we have an $N = 1$ theory in four dimensions (see e.g. [15][16][17][18][19][20]). This involves studying elliptically fibered manifolds over a 3-fold base B , which is the ‘visible’ part of the space for type IIB. The gauge symmetries are encoded in terms of the structure of the 7-brane worldvolume which is $R^4 \times S$ where S is a 4-dimensional (2-complex dimensional) subspace of B . If n parallel 7-branes coincide we get $SU(n)$ gauge symmetry, which is encoded in the elliptic fibration acquiring an A_{n-1} singularity. Similarly if we develop a D or E singularity we obtain $SO(2n)$ or E_n (or their modding out by outer automorphisms leading to $Sp(n)$, $SO(2n-1)$, F_4 , or G_2 [6][7]) gauge symmetry in four dimensions.

The situation can in general be more complicated: We could have a sublocus of S , consisting of a complex curve where the singularity gets enhanced. It could also happen that on a number of points on that enhanced symmetry loci, the symmetry may be further enhanced. In such cases the study of the theory we obtain in four dimensions is more interesting and is expected to give matter fields or more exotic objects such as tensionless strings, as has been shown in a similar context for compactification of F-theory on CY 3-folds [10][21]. In this paper we consider mainly the case without such complications. In other words we consider the case where on S we acquire some ADE singularity, and that there are no extra singularities anywhere on S . In this way we can geometrically engineer an $N = 1$ theory in $d = 4$ with ADE gauge symmetry. We can also consider outer automorphisms to get the non-simply laced groups.

Since we have no extra singularities on S we have no matter arising in a local way on S . However global aspects of S can lead to matter. The basic idea is that on $R^4 \times S$ being a D-brane worldvolume, one expects [22] a partially twisted topological field theory [23] which is twisted along S but untwisted on R^4 , with an $N = 1$ in $d = 4$. The choice of the twisting is most easily determined by the number of supersymmetries one wishes to preserve. Similar topological field theories arise upon compactification of type IIA Calabi-Yau threefolds [4] and heterotic strings on CY threefolds [24][25].

In the case at hand the theory is an eight dimensional theory with $N = 1$ supersymmetry with gauge group G . The R-symmetry is $U(1)$. We compactify on S which has $U(2)$ holonomy. The supercharges can be decomposed as

$$\mathbf{4}_s^\pm \otimes \mathbf{4}_s^\pm$$

where the first spinor is on R^4 and the second on S and the uncorrelated \pm refer to the chirality of the spinors.

Given that S is Kähler we can view the spinors as sections of $\mathcal{L} \otimes \Lambda^* T^*$, where \mathcal{L} is a square root of the canonical line bundle and $\Lambda^* T^*$ denotes all the antiholomorphic p-forms.

The supercharge carries $U(1)$ R-charge. The twisting is simply using the $U(1)$ R-symmetry to get rid of \mathcal{L} in front and make fermions transform in $\Lambda^* T^*$. This leaves us generally with one conserved supercharge for a general complex surface S corresponding to the constant function 1, and agrees with what we expect for $N = 1$ supersymmetry in $d = 4$. In eight dimensions the gauge field A and the complex scalar ϕ in the adjoint

comprise the bosonic field of the $N = 1$ multiplet. Moreover the complex scalar carries R-charge whereas the gauge bosons are neutral under R. Upon topological twisting the vector field A remains a 1-form, whereas the scalars now transform as a section of the canonical line bundle on S . The number of chiral fields in the adjoint representation we get in four dimensions from A is $h^{1,0}(S)$, which corresponds to the number of choices for the Wilson lines we can turn on. The number of adjoint chiral fields we get from ϕ is equal to $h^{2,0}(S)$, i.e., the number of zero modes of the canonical line bundle¹. In this paper we will assume that both of these numbers are zero so that we do not have any adjoint matter. We shall return to the more general case in a future publication.

For the most part in this paper we will assume that we have an *ADE* type singularity over a complex surface S which has $h^{0,1} = h^{0,2} = 0$ and that the singularity does not change over S . In this case we expect to have an $N = 1$ Yang-Mills theory in $d = 4$ of *ADE* gauge group without any matter. Note that the bare gauge coupling constant in 4 dimensions is given by

$$\frac{1}{g_4^2} = V_S \quad (2.1)$$

where V_S denotes the volume of S . This relation follows from the fact that the gauge coupling is of order 1 in 8 dimensions, and upon reduction on S picks up the volume factor of S .

The infrared dynamics in this theory is expected to involve strong coupling phenomena of confinement and gaugino condensation. Moreover it is expected that there will be $c_2(G)$ vacua, where $c_2(G)$ denotes the dual Coxeter number of the group G , corresponding to the choice of the phase of the gaugino condensation. Even though it may at first sight appear difficult to see these in this geometrical setting, it turns out to be very easy once we take one of the dimensions of space to be a compact circle of radius R . We return to these issues after we discuss the compactification on a circle to 3 dimensions.

3. $N = 2$ Yang-Mills in $d = 3$ and M-theory on CY 4-fold

If we compactify the $N = 1$ theory from $d = 4$ to $d = 3$ we obtain an $N = 2$ theory in $d = 3$. By the chain of duality in [26] the compactification of F-theory on a circle is dual to M-theory on the same elliptic Calabi-Yau where the radius of the circle R is related to the Kähler class of the elliptic fiber $k_E = \frac{1}{R}$. If we want to retain the R-dependence in

¹ There could in general be superpotentials for these adjoint zero modes dictated by geometry.

the physical quantities, we have to note that the 4-fold is an elliptic one with a singularity over the surface S .

$N = 2$ in $d = 3$ has a Coulomb branch: The Wilson line of the four dimensional gauge field along the circle as well as the dual to the vector gauge field in $d = 3$ which is a scalar, form a complex scalar field ϕ . Going to non-zero value of ϕ is realized geometrically by blowing the singularity of ADE type, and ϕ is identified with the blow up parameter.

For $N = 2$ in $d = 3$ Yang-Mills, one expects to obtain a superpotential [27] $W(\phi)$. In particular for the $SU(2)$ gauge group it was shown in [27] that a non-perturbative superpotential is generated:

$$W = \exp\left(-\frac{\phi}{g_3^2}\right)$$

where g_3 is the 3-dimensional gauge coupling constant. If this theory comes from a reduction of $N = 1$ in $d = 4$ on a circle of radius R where $1/g_3^2 = R/g_4^2$, the superpotential develops an R dependent piece. In fact it was argued in [28] that the R dependent superpotential is (in a particular normalization of ϕ)

$$W = \exp\left(-\frac{\phi}{g_3^2}\right) + \exp\left(-\frac{1}{Rg_3^2} + \frac{\phi}{g_3^2}\right)$$

This in particular is consistent with the fact that for any finite R there are 2 vacua (which solve $\partial_\phi W = 0$), in agreement with the Witten index for the $N = 1$ theory in $d = 4$. We can now ask whether we can see such superpotential directly from our geometric engineering of these field theories. We will see that not only this is possible but it also sheds light on the structure of the superpotential for $N = 2$ pure Yang-Mills in $d = 3$.

4. Generation of Superpotential

It was shown in [15] how to compute the effect of zero size instantons in M-theory and F-theory on Calabi-Yau fourfolds. This was further studied in the context of some explicit examples in [18][20][19]. This method for computation of the superpotential applies to cases where the field theory admits a phase where the non-abelian gauge symmetry is either completely broken or broken to abelian parts. This is in particular expected to be the case for the local model of $N = 2$ theories in $d = 3$ we discussed above.

The general statement in [15] is that the superpotential receives contributions from Euclidean 5-branes wrapped around non-trivial 6-cycles of the Calabi-Yau fourfold. For a

smooth 6-cycle C a necessary condition for contribution to the superpotential is that the holomorphic Euler characteristic of C be equal to 1, i.e.

$$\chi(\mathcal{O}_C) = h^{0,0} - h^{1,0} + h^{2,0} - h^{3,0} = 1$$

Moreover if $h^{1,0} = h^{2,0} = h^{3,0} = 0$ then this is a sufficient condition for the generation of the superpotential.

The case we have is a local model of an ADE singularity of the fiber over the base S . We can describe the local model of this geometry as follows. Suppose that we have a local Calabi-Yau 4-fold Y which has an ADE singularity along S and which admits a split simultaneous resolution. By this, we mean that there is assumed to exist a resolution $\pi : X \rightarrow Y$ of the ADE singularity with some additional properties. We denote the exceptional divisor by D , which is a possibly singular 3-fold. The map π restricts to D giving a map $\pi|_D : D \rightarrow S$ whose fibers are described by the appropriate ADE Dynkin diagram. The fibers are allowed to change over different points of S , i.e., we can allow for matter (or some exotic physics of enhanced symmetry points). The “split” assumption means that D is a union of r irreducible components D_i , i.e., $D = \cup_i D_i$ where r is the rank of the gauge group, i.e. the rank of the singularity at the generic point of S . The divisors D_i are smooth 3-folds. Each D_i can be viewed as a \mathbf{P}^1 bundle over S .

If we have an elliptic fibration which degenerates over S and can be resolved as above, then the fiber over S decomposes as $D \cup D'$, where D' is the closure of the complement of the exceptional set D inside the resolved elliptic fiber. Our assumptions now says that D' is a \mathbf{P}^1 bundle over S . The \mathbf{P}^1 fiber corresponding to D' forms the extra node to make the Dynkin diagram an *affine* Dynkin diagram. Note that this is consistent with the fact that the sum (with multiplicities) over all \mathbf{P}^1 's in the fiber gives the class of the elliptic curve e , which satisfies $e \cdot e = 0$, the intersection being taken within the fiber. To put in more detail, after blowing up, the elliptic fiber over S will consist of intersecting spheres which form the *affine* Dynkin diagram of ADE. For example for an A_{n-1} singularity, the elliptic fiber decomposes into a cycle of n spheres e_i which intersect two others with intersection number one.² For any ADE we have $r + 1$ classes where r corresponds to the rank of the group. We can associate each e_i with a node on the extended Dynkin diagram. Then there is a relation [29]

$$\sum_{i=1}^{r+1} a_i e_i = e \tag{4.1}$$

² Type III and IV configurations are possible as well in the respective cases of A_1 and A_2 .

where a_i denotes the Dynkin number associated with the Dynkin node e_i and e denotes the class of the elliptic fiber.

Note that blowing up the singularity can only be done in M-theory, because from the M-theory viewpoint, equation (4.1) implies that if the singularity is blown up the Kähler class of the elliptic fiber is not zero. The F-theory limit is obtained by pushing the Kähler class of the elliptic fiber to zero size, which means turning off the blow up modes. This is of course expected because only in the M-theory case we have a Coulomb phase (corresponding to the expectation value for additional scalar in the gauge multiplet). $N = 1$ pure Yang-Mills theory in 4-dimensions has no moduli.

To compute the 5-brane corrections to the superpotential we have to identify the complex 3-surfaces with holomorphic Euler characteristic 1. There are $r+1$ such instantons and they are simply in one to one correspondence with the complex 3-folds consisting of the e_i sphere over S . We see this as follows.

Recall that for any complex manifold N , the cohomology group $H^i(N, \mathcal{O}_N)$ can be identified with the i^{th} $\bar{\partial}$ -cohomology group of N . If there is a fibration $\phi : N \rightarrow P$ of complex manifolds N, P whose fibers are smooth and have vanishing higher $\bar{\partial}$ -cohomology groups, then it is not too hard to see that the $\bar{\partial}$ -cohomology groups of N and P must coincide. As a consequence, we conclude in this case that $\chi(\mathcal{O}_N) = \chi(\mathcal{O}_P)$.

We want to apply this argument to the fibrations $D_i \rightarrow S$ and $D' \rightarrow S$ arising from our elliptic fibration. Although D_i , D' , and S are smooth, the fibers may be singular, so the argument does not apply. But there is a simple generalization. Returning to the general situation $\phi : N \rightarrow P$, let's remove the condition that the fibers are smooth with vanishing higher $\bar{\partial}$ -cohomology, and replace it by the conditions

$$H^i(\phi^{-1}(p), \mathcal{O}_{\phi^{-1}(p)}) = 0 \quad \forall i > 0, \quad p \in P$$

and the condition that the only holomorphic functions on the fibers are constants. We again can conclude that $H^{i,0}(N) \simeq H^{i,0}(P)$ for all i and therefore $\chi(\mathcal{O}_N) = \chi(\mathcal{O}_P)$.³

³ This can be established using the Leray spectral sequence [30]

$$H^p(P, R^q \phi_*(\mathcal{O}_N)) \Rightarrow H^{p+q}(N, \mathcal{O}_N)$$

which degenerates as our assumptions show that $R^q \phi_*(\mathcal{O}_N) = 0$ for $q > 0$. Furthermore, our assumptions also imply that $R^0 \phi_*(\mathcal{O}_N) = \phi_*(\mathcal{O}_N) = \mathcal{O}_P$. We then conclude that

$$H^i(N, \mathcal{O}_N) \simeq H^i(P, \mathcal{O}_P).$$

for all i . This implies that $h^{i,0}(N) = h^{i,0}(P)$, establishing the desired result.

Returning to the fibrations $D_i \rightarrow S$ and $D' \rightarrow S$, and recalling that all of the fibers arise as exceptional curves of ADE resolutions, we only have to compute the holomorphic cohomology groups of these exceptional curves. The holomorphic cohomology groups for $i > 1$ vanish since the fibers are curves. It remains only to compute H^0 and H^1 .

If $g : \tilde{M} \rightarrow M$ is the minimal resolution of any rank r ADE surface singularity $p \in M$, then the exceptional curve $C = g^{-1}(p)$ is a union of r copies of \mathbf{P}^1 intersecting according to the Dynkin diagram. The equations vanishing on p pull back to equations defining C , and the components e_j of C occur with multiplicity equal to the corresponding Dynkin number a_j . We can express this as $C = \sum a_j e_j$.⁴ Explicit equations which make these multiplicities plain are given for example in the appendix of [31]. Each curve C is known to be connected and rational (so that $C \cdot C = -2$), which implies that $H^1(C, \mathcal{O}_C) = 0$ and that C has no nonconstant holomorphic functions.

We can now conclude that $\chi(\mathcal{O}_{D_j}) = 1$ and $h^{i,0}(D_j) = 0$ for $i > 0$, and that $\chi(\mathcal{O}_{D'}) = 1$ and $h^{i,0}(D') = 0$ for $i > 0$.

In a local model for the elliptic fibration, i.e. a 4-fold neighborhood X of $D \cup D'$ mapping to a 4-fold neighborhood of S , the only irreducible compact 3-folds contained in X are precisely the D_j and D' . There are exactly $r + 1$ such 3-folds, and they all have $\chi(\mathcal{O}) = 1$ and vanishing higher holomorphic cohomology. Therefore in this situation, we always get $r + 1$ instantons.

We thus obtain

$$W = \sum_{i=1}^{r+1} \exp(-V_i) \quad (4.2)$$

where V_i is the complexified Kähler class corresponding to the S_i which is the total space of e_i over S . For $i = 1, \dots, r$ this can be viewed as

$$V_i = V_S V_{e_i} = \frac{1}{g_3^2} \phi_i$$

where we identify the blow up parameter ϕ_i with the volume of e_i . In the limit where the Kähler class of the elliptic fiber becomes large, the relevant part of the singularity is captured by the ordinary Dynkin diagram. Note that the contribution of the extra class corresponding to the extra node of the Affine Dynkin diagram to the superpotential can

⁴ More precisely, C has the structure of a scheme. It is possible for schemes to have nonconstant holomorphic functions on compact connected sets. This is why the condition on the fibers of ϕ above were required.

be computed from (4.1). Note that the volume of the Euclidean fivebrane corresponding to the extended node is $V_S V_{T^2 - \sum_i a_i e_i}$ and gives rise to the superpotential

$$\exp(-V_S V_{T^2 - \sum_i a_i e_i}) = \exp(-V_S V_{T^2}) \exp(V \sum_i a_i e_i) = \exp(-\frac{1}{R g_3^2}) \exp(\sum_i a_i V_i)$$

where we used the relation between the radius R of F-theory compactification and the Kähler class of the T^2 fiber in the M-theory. Thus putting the contribution of all the $r + 1$ instantons together we find

$$W = \sum_{i=1}^r \exp(-V_i) + \gamma \exp(\sum_{i=1}^r a_i V_i) \quad (4.3)$$

where

$$\gamma = \exp(-1/R g_3^2)$$

For the case of $SU(2)$ the result above agrees with the results obtained recently in [28]. The generalization we have found here to all the gauge groups is extremely simple and suggestive. In fact the superpotential (4.3) is exactly the same as the potential for affine Toda theory for ADE; this link suggests that this theory may in some sense be integrable. The radius dependent factor γ goes from 1 to 0 as we go from $R = \infty$ to $R = 0$. In the $R = 0$ limit the potential for the affine Toda theory goes over to that of the Toda theory. Thus the connection from 3 dimensions to 4 dimensions amounts to replacing the Toda superpotential with the affine Toda version. This is similar to the integrable structure found recently [32] in the context of $N = 2$ theories in going from $d = 4$ to $d = 5$ in which a prepotential of an integrable system was replaced by the relativistic version in going up one dimension.

This connection with integrable structure also suggests that upon further compactification on a circle the theory may go over to the *integrable* $N = 2$ supersymmetric theories with affine Toda superpotential, studied in [33]. It would be interesting to explore this connection.

Let us subject (4.3) to a simple test. If we have an $N = 1$ pure Yang-Mills in 4 dimensions with gauge group G , we expect to have $c_2(G)$ inequivalent vacua. Note that the limit of $d = 4$ is obtained in the above by setting $\gamma = 1$. In fact for any non-zero value of γ we expect to find $c_2(G)$ vacua given by the choice of phase of the gaugino condensation. This follows from the fact that in the presence of an instanton there are $2c_2(G)$ gaugino zero modes and factorization arguments (with the assumption of mass gap) leads us to

$c_2(G)$ choices for vacua depending on the phase of the gaugino condensate $\langle \lambda^2 \rangle$. To see if this is in accord with our results we need to find solutions to $\partial_{V_i} W = 0$ in (4.3). We obtain

$$\frac{\exp(-V_i)}{a_i} = \gamma \exp(\sum a_i V_i) = \beta$$

which implies that

$$\gamma \prod_{i=1}^r (a_i \beta)^{-a_i} = \beta$$

Since we have the group theoretic identity $\sum_{i=1}^r a_i = c_2(G) - 1$ this leads to

$$\beta^{c_2(G)} = A$$

where $A = \gamma \prod_i a_i^{-a_i}$. This equation has $c_2(G)$ solutions for β as expected. In fact we can identify the configuration of vacua we have found with the choices for discrete 't Hooft fluxes [34] along the compact direction. After all, ϕ_i represents electric and magnetic Wilson lines along the compact direction. This is also in the same spirit as the computation of the number of ground state vacua in this case [35].

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